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Pullback attractors of 2D Navier-Stokes-Voigt equations with delay on a non-smooth domain

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Abstract

Under suitable hypotheses on the continuous delay, distributed delay, and the initial data in this paper, the large-time behavior for the 2D Navier-Stokes-Voigt equations with continuous delay and distributed delay on the Lipschitz domain is studied. The existence of pullback attractors in the non-smooth domain was obtained via verifying some pullback dissipation and asymptotical compactness for the continuous process.

Keywords: Navier-Stokes-Voigt equation; continuous delay; distributed delay; pullback attractors; Lipschitz domain

1 Introduction

The Navier-Stokes equations essentially constitute a description of hydrodynamical systems, and the well-posedness and large-time behavior of solutions to the Navier-Stokes equations have received very much attention in the understanding of fluid motion and turbulence. For the 2D case, Ladyzhenskaya [1] solved the uniqueness of the global smooth solutions. Based on the well-posedness of the incompressible Navier-Stokes equations (NSE), the infinite dimensional dynamical systems also was investigated. Till now there are many interesting results (the existence of global attractor, uniform attractors, pullback attractors, the structure and dimensions of attractors), the 2D general case can be found in [2–7], and these results were studied in the regular domain. But the well-posedness of global solutions for the Navier-Stokes equations in the non-regular domain (such as the Lifchitz domain) is a difficult problem, and there are fewer corresponding results. For the 2D incompressible Navier-Stokes equation, Brown *et al.* [8] constructed a stream function which will be used later in our paper to solve the non-homogeneous boundary value problems and gave the existence, dimension of global attractor.

Very recently, using an approximation of the Navier-Stokes equation (such as the Navier-Stokes-Voigt equation (NSVE)) to study the existence of attractor of classical NSE has become a topic generally focused on. The Navier-Stokes-Voigt system is the classical Navier-Stokes system with strong damping which models the dynamics of a Kelvin-Voigt viscoelastic incompressible fluid and was first introduced by Oskolkov [9] as a model of the motion of linear viscoelastic fluids. More results about the well-posedness, the existence of attractor and the infinite dimensional systems, we can see [10, 11]. Also we mention the large-time behavior for the NSV system with delay, especially with both the continuous delay and the distributed delay, which are similar to the memory in elastic system.

The Navier-Stokes equations with delay was studied recently. In 2002, Krasovskii [12] constructed the Navier-Stokes equations with delay and obtained the well-posedness. Barbu and Sritharan [13] established the existence and uniqueness of weak solutions to the Navier-Stokes equations with the forcing term containing delay in 2003. Taniguchi [14] established the existence of absorbing sets of the non-autonomous Navier-Stokes equations with continuous delay in 2005. Caraballo and Real [15–17] studied the Navier-Stokes system with continuous delay and distributed delay and obtained the existence of global and pullback attractors for autonomous and non-autonomous cases, respectively. Marín-Rubio and Real [18] investigated the Navier-Stokes equation with delay on some unbounded domain with the Poincaré inequality and obtained the pullback attractors. Garrido-Atienza and Marín-Rubio [19] also studied the Navier-Stokes equations with delay on an unbounded domain and proved some results on the existence and uniqueness of solutions. In 2014, García-Luengo *et al.* [20] studied the 2D Navier-Stokes system with the convective term and external force both containing delay and proved the existence of pullback attractors.

In this paper we consider the existence of pullback attractor of the 2D incompressible Navier-Stokes-Voigt equation with continuous delay and distributed delay on the Lipschitz domain,

$$\begin{cases} \frac{du}{dt} - \nu\Delta u - \alpha^2\Delta u_t + (u\cdot\nabla)u + \nabla p \\ \quad = f(t-\rho(t), u(t-\rho(t))) + \int_{-h}^0 G(s, u(t+s))\,ds, & (x,t)\in\Omega_\tau, \\ \operatorname{div} u = 0, & (x,t)\in\Omega_\tau, \\ u(t,x)|_{\partial\Omega} = \varphi, \qquad \varphi\cdot n = 0, & (x,t)\in\partial\Omega_\tau, \\ u(\tau,x) = u_\tau(x), & x\in\Omega, \\ u(t,x) = \phi(t-\tau,x), & (x,t)\in\Omega_{\tau h}, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a Lipschitz domain, $\Omega_\tau = \Omega\times(\tau,+\infty)$, $\partial\Omega_\tau = \partial\Omega\times(\tau,+\infty)$, $\Omega_{\tau h} = \Omega\times(\tau-h,\tau)$, $\tau\in\mathbb{R}$ is the initial time. ν is the kinematic viscosity of the fluid, u is the unknown velocity field of the fluid, p the pressure, and $\alpha > 0$ a length scale parameter characterizing the elasticity of the fluid, $f(t-\rho(t), u(t-\rho(t)))$ the external force term which contains a memory effect during a fixed interval of time of length $h > 0$, and $\rho(t)$ a given adequate delay function. Moreover, the inhomogeneous boundary function φ satisfies $\varphi\in L^\infty(\partial\Omega)$, $\int_{-h}^0 G(s,u(t+s))\,ds$ is another external force with some hereditary characteristic, and ϕ the initial state of delay in $(-h,0)$ where $h > 0$ is a constant.

Inspired by [8, 17], we shall use the background function for the Stokes problem and some pullback dissipation, and asymptotical compactness for the continuous process via the embedding theorem to achieve the pullback attractor. The main features of our present work are summarized as follows:

(1) There are many results as regards the existence of attractors of NS (NSV) equations with delay on the regular domain (such as [15–17]), many conclusions concerning the NS (NSV) equations without delay on the non-regular domain (such as the Lipschitz domain in [8]), but less work about the existence of attractors of NSV equations with continuous delay and distributed delay on the Lipschitz domain.

(2) Since our problem is studied on the Lipschitz domain (not the regular domain), we use Hardy's inequality and smooth approximation functions to deal with the non-regular boundary. Defining some suitable topology spaces for the solutions and dealing with each delay term to get some *a priori* estimate on non-smooth domain for the pullback absorbing sets and asymptotical compactness, we conclude to the existence of a pullback attractor for (1.1).

The structure of this paper is the following. In Section 2, some preliminaries are given which will be used in sequel. The existence and uniqueness of solution for our problem are derived in Section 3. In Sections 4 and 5, the existence of pullback attractors for the problem (1.1) is derived in the appropriate topology space.

2 Preliminaries

Denote $E := \{u | u \in (C_0^\infty(\Omega))^2, \operatorname{div} u = 0\}$, H is the closure of the set E in $(L^2(\Omega))^2$ topology, $|\cdot|$ and $(\cdot,\cdot)$ represent the norm and inner product in H, respectively, *i.e.*,

$$|u| = \left(\int_\Omega |u|^2\,dx\right)^{1/2}, \qquad (u,v) = \sum_{j=1}^{2}\int_\Omega u_j(x)v_j(x)\,dx, \quad \forall u,v \in \left(L^2(\Omega)\right)^2. \tag{2.1}$$

V is the closure of the set E in $(H^1(\Omega))^2$ topology, and $\|\cdot\|$ and $((\cdot,\cdot))$ denote the norm and inner product in V, respectively, *i.e.*,

$$\|u\| = \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}, \qquad \bigl((u,v)\bigr) = \sum_{i,j=1}^{2}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\,dx, \quad \forall u,v \in V. \tag{2.2}$$

P is the Helmholtz-Leray orthogonal projection in $(L^2(\Omega))^2$ onto the space H, $A := -P\Delta$ is the Stokes operator, the sequence $\{\omega_j\}_{j=1}^\infty$ is an orthonormal system of eigenfunctions of A, and $\{\lambda_j\}_{j=1}^\infty$ $(0 < \lambda_1 \le \lambda_2 \le \cdots)$ is the eigenvalue of A corresponding to the eigenfunction $\{\omega_j\}_{j=1}^\infty$. We can define the power A^s for $s \in \mathbb{C}$ as follows:

$$A^s f = \sum_j \lambda_j^s a_j \omega_j, \quad s \in \mathbb{C}, j \in \mathbb{R}, f = \sum_j a_j \omega_j, \tag{2.3}$$

$$D\bigl(A^s\bigr) = \bigl\{f : A^s f \in H\bigr\} = \left\{f = \sum_j a_j\omega_j : \sum_j \lambda_j^{\operatorname{Re} Z}|a_j|^2 < +\infty\right\}, \tag{2.4}$$

$D(A^s)$ is the domain of A^s, and we still denote the closure of E in $D(A^s)$ by $D(A^s)$. The norm of $D(A^{\frac{s}{2}})$ is written as $\|u\|_s$, and A^s has the following properties (see [8]):

$$\int_\Omega \frac{|A^\alpha u|^2}{\operatorname{dist}(x,\partial\Omega)}\,dx \le C_0\int_\Omega \left|A^{\alpha+\frac{1}{4}}u\right|^2 dx, \quad \forall u \in D\bigl(A^{\alpha+\frac{1}{4}}\bigr), \tag{2.5}$$

$$\|u\|_{L^4} \le C_1\left|A^{\frac{1}{4}}u\right|, \quad \forall u \in D\bigl(A^{\frac{1}{4}}\bigr), \tag{2.6}$$

where V is a Hilbert space, and $\|v\| = |\nabla v|$. Clearly, $V \hookrightarrow H \equiv H' \hookrightarrow V'$, H' and V' are dual spaces of H and V, respectively, where the injection is dense and continuous. The norm $\|\cdot\|_*$ and $\langle\cdot\rangle$ denote the norm in V' and the dual product between V and V', respectively.

The bilinear form operator and trilinear form operator are defined as follows (see [21]):

$$B(u,v) := P\big((u\cdot\nabla)v\big), \quad \forall u,v\in E, \tag{2.7}$$

$$b(u,v,w) = \big(B(u,v),w\big) = \sum_{i,j=1}^{2}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j\,dx, \tag{2.8}$$

where $B(u,v)$ is a linear continuous operator from V to V' which maps W into H, and $b(u,v,w)$ satisfies

$$\begin{cases} b(u,v,v)=0, & \forall u,v,w\in V,\\ b(u,v,w)=-b(u,w,v), & \forall u,v,w\in V,\\ |b(u,v,w)|\le C|u|^{\frac12}\|u\|^{\frac12}\|v\|^{\frac12}|Av|^{\frac12}|w|, & \forall u\in V, v\in D(A), w\in H,\\ |b(u,v,u)|\le C|u|^{\frac12}|Au|^{\frac12}\|v\||w|, & \forall u\in D(A), v\in V, w\in H,\\ |b(u,v,w)|\le C|u|\|v\||w|^{\frac12}|Aw|^{\frac12}, & \forall u\in H, v\in V, w\in D(A),\\ |b(u,v,w)|\le C|u|^{\frac12}\|u\|^{\frac12}\|v\||w|^{\frac12}\|w\|^{\frac12}, & \forall u\in V, v\in D(A), w\in H, \end{cases} \tag{2.9}$$

and we introduce some useful inequalities, lemmas, and definitions.

Young's inequality:

$$ab\le\frac{\varepsilon}{p}a^p+\frac{1}{q\varepsilon^{\frac{1}{p-1}}}b^q, \quad q=\frac{p}{p-1}, 1<p<\infty, \forall a,b,\varepsilon>0. \tag{2.10}$$

The Poincaré inequality:

$$|u|\le\lambda_1^{-\frac12}\|u\|, \quad \forall u\in V. \tag{2.11}$$

The Gagliardo-Nirenberg interpolation inequality:

$$\big|A^{1/2}u\big|^2\le C_2\big|A^{1/4}u\big|\big|A^{3/4}u\big|, \quad \forall u\in D\big(A^{3/4}\big). \tag{2.12}$$

Hardy's inequality:

$$\int_\Omega\frac{|u(x)|^2}{[\operatorname{dist}(x,\partial\Omega)]^2}\,dx\le C_3\int_\Omega\big|\nabla u(x)\big|^2\,dx, \quad \forall u\in V. \tag{2.13}$$

Definition 2.1 Let X and Y be Banach spaces, $X\subset Y$, we say that X is compactly embedded in Y, written as

$$X\hookrightarrow\hookrightarrow Y,$$

provided

(i) $\|x\|_Y\le C\|x\|_X$ $(x\in X)$ for some constant C;
(ii) each bounded sequence in X is precompact in Y.

Lemma 2.1 *Let $X=H, V$ or V', then $\|Pu\|_X\le\|u\|_X$, and $Pu\to u$ in X.*

Proof See, *e.g.*, [22] or [21]. □

Lemma 2.2 *Let $X \subset\subset H \subset Y$ be Banach spaces, and X is reflective. If u_n is a uniformly bounded sequence in $L^2(\tau, T; Y)$, and there exists $p > 1$ such that $\frac{dv_n}{dt}$ is uniformly bounded in $L^p(\tau, T; Y)$, then u_n has a strong convergence subsequence in $L^2(\tau, T; H)$.*

Proof See, *e.g.*, [22] or [21]. □

Lemma 2.3 (The Gronwall inequality) *Let g, h, and y all be locally integrable functions in $(t_0, +\infty)$ satisfying*

$$\frac{dy}{dt} \leq gy + h, \quad \forall t \geq t_0,$$

and $\frac{dy}{dt}$ is locally integrable, then we have

$$y(t) \leq y(t_0)e^{\int_{t_0}^{t} g(\tau)\,d\tau} + \int_{t_0}^{t} h(s)e^{-\int_{t_0}^{t} g(\tau)\,d\tau}\,ds, \quad \forall t \geq t_0.$$

Proof See, *e.g.*, [21]. □

Lemma 2.4 (The generalized Arzelà-Ascoli theorem) *Let $\{f_r(\theta) : \gamma \in \Gamma\} \subset C = C([-r, 0]; X)$ is equicontinuous, and for all $\theta \in [-r, 0]$, $\{f_r(\theta) : \gamma \in \Gamma\}$ is relatively compact in $C([-r, 0]; X)$.*

Proof See, *e.g.*, [22]. □

Definition 2.2 Let X be a metric space, the set class $\{U(t, \tau)\}$ $(-\infty < \tau \leq t < +\infty) : X \to X$ is called a processes in X, if

(i) $U(\tau, \tau)x = x$, $\tau \in R$, $\forall x \in X$;
(ii) $U(t, \tau) = U(t, s)U(s, \tau)$, $\forall \tau \leq s \leq t$, $\tau \in R$.

Let $\mathcal{P}(X)$ denote all the family of nonempty subsets of X, and $\mathcal{D}$ the class of all families $\hat{D} = \{D(t)|t \in R\} \subset \mathcal{P}(X)$.

Definition 2.3 The processes class $\{U(\cdot, \cdot)\}$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in R$, $\hat{D} \in \mathcal{D}$ and $\tau_n \to -\infty$, $x_n \in D(\tau_n)$, the sequence $\{U(t, \tau_n)x_n\}$ possesses a convergent subsequence.

Definition 2.4 A family $B = \{B(t)|t \in R\} \in \mathcal{P}(X)$ is said to be pullback $\mathcal{D}$-absorbing if for each $t \in R$ and $\hat{D} \in \mathcal{D}$, there exists $\tau_0(t, \hat{D}) \leq t$ such that

$$U(t, \tau)D(\tau) \subset B(t), \quad \forall \tau \leq \tau_0(t, \hat{D}).$$

Definition 2.5 A family $\hat{A} = \{A(t)|t \in R\} \in \mathcal{P}(X)$ is said to be a global pullback $\mathcal{D}$-attractor with respect to the processes $\{U(\cdot, \cdot)\}$, if

(i) $A(t)$ is compact for any $t \in R$;

(ii) $\hat{A}$ is pullback $\mathcal{D}$-attracting, *i.e.*,

$$\forall \hat{D} \in \mathcal{D}, t \in R, \quad \lim_{\tau \to -\infty} \operatorname{dist}\big(U(t,\tau)D(\tau), A(t)\big) = 0,$$

where $\operatorname{dist}(C_1, C_2)$ denotes the Hausdorff semi-distance between C_1 and C_2 defined as $\operatorname{dist}(C_1, C_2) = \sup_{x\in C_1} \inf_{y\in C_2} d(x,y)$ for $C_1, C_2 \subset X$;

(iii) $\hat{A}$ is invariant, *i.e.*, for all $-\infty < \tau \le t < +\infty$, we have $U(t,\tau)A(\tau) = A(t)$.

Definition 2.6 We claim that $A(t) = \overline{\bigcup_{\hat{D}\in\mathcal{D}} \Lambda(\hat{D}, t)}$, $t \in R$, where $\Lambda(\hat{D}, t)$ is defined as

$$\Lambda(\hat{D}, t) = \bigcap_{s\le t} \biggl(\overline{\bigcup_{\tau \le s} U(t,\tau)D(\tau)}\biggr), \quad \forall \hat{D} \in \mathcal{D}.$$

Theorem 2.1 *Let the process* $\{U(t,\tau)\}$ *be continuous and pullback* $\mathcal{D}$*-asymptotically compact, and let there exist* $\hat{B} \in \mathcal{D}$ *which is pullback* $\mathcal{D}$*-absorbing with respect to* $\{U(t,\tau)\}$. *Then the family* $\hat{A} = \{A(t) | t \in R\} \subset \mathcal{P}(X)$, $A(t) = \Lambda(\hat{B}, t)$, $t \in R$ *is a global pullback* $\mathcal{D}$*-attractor which is minimal in the sense that if* $\hat{C} = \{C(t) | t \in R\} \subset \mathcal{P}(X)$ *is closed and* $\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)B(\tau), C(t)) = 0$, *then* $A(t) \subset C(t)$.

Proof See, *e.g.*, [23]. □

3 Existence of solutions, uniqueness, and continuity results

3.1 Stream function

First we introduce a stream function ψ which solves the Stokes system (see [8])

$$\begin{cases} -\triangle u + \nabla q = 0, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = \varphi & \text{a.e. on } \partial\Omega \text{ in the sense of nontangential convergence,} \end{cases} \tag{3.1}$$

and ψ satisfies

$$\sup_{x\in\Omega} \bigl|\psi(x)\bigr| + \sup_{x\in\Omega} \bigl|\nabla\psi(x)\bigr| \operatorname{dist}(x, \partial\Omega) \le C_4 \|\varphi\|_{L^\infty(\partial\Omega)}, \tag{3.2}$$

$$\bigl\| |\nabla\psi| \operatorname{dist}(\cdot, \partial\Omega)^{1-\frac{1}{p}} \bigr\|_{L^p(\Omega)} \le C_5 \|\varphi\|_{L^p(\partial\Omega)}, \quad 2 \le p \le \infty. \tag{3.3}$$

It follows that

$$\|\psi\|_{L^\infty(\Omega)} \le C_4 \|\varphi\|_{L^\infty(\partial\Omega)}. \tag{3.4}$$

Let $\varepsilon \in (0, c \cdot \operatorname{diam}(\Omega))$ be a constant to be determined later, and $\eta_\varepsilon \in C_0^\infty(\mathbb{R}^2)$ such that

$$\begin{cases} \eta_\varepsilon = 1, & \text{in } \{x \in R^2 | \operatorname{dist}(x, \partial\Omega) \le C_1'\varepsilon\}, \\ \eta_\varepsilon = 0, & \text{in } \{x \in R^2 | \operatorname{dist}(x, \partial\Omega) \ge C_2'\varepsilon\}, \\ 0 \le \eta_\varepsilon \le 1, & \text{otherwise,} \end{cases} \tag{3.5}$$

and

$$\left|\nabla^{\alpha}\eta_{\varepsilon}\right| \leq C'_{\alpha}/\varepsilon^{|\alpha|}, \tag{3.6}$$

where η_{ε} is in the form $h(\frac{\rho(x)}{\varepsilon})$, h is a standard bump function, and $\rho \in C^{\infty}$ is a regularized distance bump function to $\partial\Omega$.

We also know

$$\operatorname{div}\psi = 0, \qquad x \in \Omega; \qquad \psi = u, \qquad x \in \left\{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < C'_1\varepsilon\right\}, \tag{3.7}$$

$$\psi = \varphi, \quad \text{on } \partial\Omega \text{ in the sense of nontangential convergence}, \tag{3.8}$$

and

$$\operatorname{Supp}\psi \subset \left\{x \in \bar{\Omega}; \operatorname{dist}(x, \partial\Omega) < C'_2\varepsilon\right\}. \tag{3.9}$$

Lemma 3.1 *Assume* ψ *satisfies* (3.7)-(3.9), *then we have*

$$\Delta\psi = \nabla(q\eta_{\varepsilon}) + F, \tag{3.10}$$

where

$$\|F\|_{L^2(\Omega)} \leq C/\varepsilon^{\frac{3}{2}}\|\varphi\|_{L^2(\partial\Omega)}, \qquad \nabla q = \Delta u, \tag{3.11}$$

$$F = 0, \quad \textit{if } x \in \left\{x \mid \operatorname{dist}(x, \partial\Omega) < C'_1\varepsilon \textit{ or } \operatorname{dist}(x, \partial\Omega) > C'_2\varepsilon\right\}. \tag{3.12}$$

3.2 Assumptions and abstract equation

For any $t \in (\tau, T)$, we define $u : (\tau - h, T) \to (L^2(\Omega))^2$, and u_t is a function defined on $(-h, 0)$ satisfying $u_t = u(t+s)$, $s \in (-h, 0)$. Let

$$C_H = C^0\big([-h,0];H\big), \qquad C_{D(A^{3/4})} = C^0\big([-h,0];D\big(A^{3/4}\big)\big), \qquad C_V = C^0\big([-h,0];V\big)$$

be three Banach spaces with the norms

$$\|u\|_{C_H} = \sup_{\theta\in[-h,0]}\left|u(t+\theta)\right|, \qquad \|u\|_{C_V} = \sup_{\theta\in[-h,0]}\left\|u(t+\theta)\right\|,$$

$$\|u\|_{C_{D(A^{3/4})}} = \sup_{\theta\in[-h,0]}\left\|u(t+\theta)\right\|_{3/4},$$

respectively, and

$$L^2_H = L^2(-h,0;H), \qquad L^2_V = L^2(-h,0;V),$$

$$L^{\infty}_H = L^{\infty}(-h,0;H), \qquad L^{\infty}_V = L^{\infty}(-h,0;V).$$

The problem (1.1) can be written as the abstract form

$$\begin{cases} \frac{du}{dt} + \nu Au + \alpha^2 Au_t + B(u) = f_{\rho}(u) + g(t,u_t), \\ u(\tau) = u_{\tau}, \qquad u(t) = \phi(t-\tau), \quad t \in (\tau - h, \tau), \end{cases} \tag{3.13}$$

where $f_\rho(u) = f(t-\rho(t), u(t-\rho(t)))$, $g(t,u_t) = \int_{-h}^0 G(t,s,u(t+s))\,ds$, and it satisfies

(a) $\forall \xi \in C_H$, $t \in \mathbb{R} \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is measurable and $g(t,0) = 0$, $\forall t \in \mathbb{R}$;

(b) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$, $\xi, \eta \in C_H$,

$$\left|g(t,\xi) - g(t,\eta)\right| \le L_g \|\xi - \eta\|_{C_H};$$

(c) $\exists m_0 \ge 0$, $C_g > 0 : \forall m \in [0,m_0]$, $\tau \le t$, $u,v \in C^0([\tau-h,t];H)$,

$$\int_\tau^t e^{ms}\left|g(s,u_s) - g(s,v_s)\right|^2 ds \le C_g^2 \int_{\tau-h}^t e^{ms}\left|u(s) - v(s)\right|^2 ds;$$

(d) $\rho : [0,\infty) \to [0,h]$, $|\frac{d\rho}{dt}| \le M < 1$;

(e) $f(t,u)$ satisfies the Lipschitz condition with respect to u: $\exists L(\beta) > 0$ such that

$$\left|f(t,u) - f(t,v)\right| \le L(\beta)|u-v|;$$

(f) $\exists a > 0, b > 0$ such that $|f(t,u)|^2 \le a|u|^2 + b$;

(g) $\nu > \frac{3C_g}{\lambda_1}$;

(h) under the conditions (a)-(g), $\exists K_1 > 0$, and let

$$\nu > \frac{6C_1^4}{\nu\lambda_1}K_1^2 + \frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{2C_2C_g}{3\nu\lambda_1} + \frac{2C_g}{\lambda_1^{\frac{3}{2}}}.$$

Let $v = u - \psi$, (1.1) can be reduced to the following system:

$$\begin{cases} \frac{\partial v}{\partial t} - \nu\Delta v - \alpha^2\Delta v_t + (v\cdot\nabla)v + (v\cdot\nabla)\psi \\ \qquad + (\psi\cdot\nabla)v + \nabla(p - \nu q\eta_\varepsilon) \\ \quad = \bar{f} - (\psi\cdot\nabla)\psi + g(t, v_t + \psi), & (x,t) \in \Omega_\tau, \\ \operatorname{div} v = 0, & (x,t) \in \Omega_\tau, \\ v = 0, & (x,t) \in \partial\Omega_\tau, \\ v(\tau,x) = v_\tau(x), & x \in \Omega, \\ v(t,x) = \phi(t-\tau,x) - \psi(x) = \eta(t-\tau,x), & (x,t) \in \Omega_{\tau h}, \end{cases} \tag{3.14}$$

where $\bar{f} = f_\rho(v+\psi) + \nu F$, $g(t,v_t+\psi) = \int_{-h}^0 G(s, v(t+s)+\psi)\,ds$, $\phi \in L_V^2 \cap L_H^\infty$.

Let $v_0 \in H$, $\eta \in L_H^2$, we consider the equivalent abstract system of (3.14)

$$\begin{cases} \frac{dv}{dt} + \nu Av + \alpha^2 Av_t + B(v) + R(v) = P\bar{f} - B(\psi) + g(t, v_t + \psi), \\ v(\tau) = v_\tau, \\ v(t) = \eta(t-\tau), \end{cases} \tag{3.15}$$

where $R(v) = B(v,\psi) + B(\psi,v)$, which is also a linear continuous operator from V into V' and maps W into H (see [21]).

Definition 3.1 Let $u_\tau, f \in H$, $\varphi \in L^\infty(\partial\Omega)$ and $\varphi \cdot n = 0$ on $\partial\Omega$, u is called a weak solution of the problem (1.1) provided

(i) $u \in C([\tau - h, T]; V)$, $u(\cdot, \tau) = u_\tau$, and $du/dt \in L^2([\tau, T]; V')$;
(ii) $\forall v \in C_0^\infty(\Omega)$ with $\operatorname{div} v = 0$, we get

$$\begin{aligned} &\frac{d}{dt}\langle u, v\rangle - \nu\langle u, \Delta v\rangle - \alpha^2 \frac{d}{dt}\langle u, \Delta v\rangle - \int_\Omega \sum_{i,j=1}^{2} u^i u^j \frac{\partial v^i}{x_j}\,dx \\ &\quad= \langle f, v\rangle + \left\langle \int_{-h}^{0} G\big(s, u(t+s)\big)\,ds, v\right\rangle; \end{aligned}$$

(iii) $\exists \psi \in C^2(\Omega) \cap L^\infty(\Omega)$, $q \in C^1(\Omega)$ and $g \in L^2(\Omega)$ such that

$$\begin{cases} \triangle\psi = \nabla q + g, & \text{in } \Omega, \\ \operatorname{div}\psi = 0, & \text{in } \Omega, \\ \psi = \varphi & \text{on } \partial\Omega, \end{cases}$$

where we assume that ψ obtain its boundary values in sense of non-tangential convergence and $u - \psi \in L^2([\tau, T]; V)$.

3.3 Existence of solutions and uniqueness

We shall give the main result in this section.

Theorem 3.1 *Let* $v_\tau \in V$, $\eta \in L^2_H$, *and the assumptions* (a)-(h) *hold, then there exists a unique global weak solution of* (3.15) *which satisfies*

$$v(t) \in L^\infty(\tau, T; V) \cap L^2(\tau, T; V),$$

and $\frac{dv}{dt}$ *is uniformly bounded in* $L^2(\tau, T; V')$.

Proof We first use the standard Faedo-Galerkin method to establish the existence of a solution to (3.15).

Fix $n \geq 1$, we define an approximate solution v_n to (3.15) as $v_n(t) = \sum_{j=0}^{n} a_{nj}(t) w_j$, which satisfies

$$\begin{cases} \frac{dv_n}{dt} + \nu A v_n + \alpha^2 A v_{nt} + B(v_n) + R(v_n) = P_n \bar{f} - B(\psi) + g(t, v_{nt} + \psi), \\ v_n(\tau) = v_{n\tau}, \\ v_n(t) = \eta_n(t - \tau), \quad t \in (\tau - h, \tau). \end{cases} \tag{3.16}$$

We also denote $f_n = f(t, v_n(t) + \psi)$, $f_{n\rho} = f(t - \rho(t), v_n(t - \rho(t)) + \psi)$, and $g_n = g(t, v_n(t) + \psi)$. Multiplying (3.16) by v_n, we have

$$\begin{aligned} &\left(\frac{dv_n}{dt}, v_n\right) + \nu(A v_n, v_n) + \alpha^2(A v_{nt}, v_n) + \big(B(v_n), v_n\big) + \big(R(v_n), v_n\big) \\ &\quad= \langle P_n \bar{f}, v_n\rangle - \big(B(\psi), v_n\big) + \langle g_n, v_n\rangle \end{aligned} \tag{3.17}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(|v_n|^2+\alpha^2\|v_n\|^2\right)+\nu\|v_n\|^2\le\left|b(v_n,v_n,v_n)\right|+\left|b(\psi,v_n,v_n)\right|+\left|b(v_n,\psi,v_n)\right|$$
$$+\left|\langle P_n\bar{f},v_n\rangle\right|+\left|\left(B(\psi),v_n\right)\right|+\left|\langle g_n,v_n\rangle\right|. \tag{3.18}$$

We estimate each term on the right side of (3.18) in the following.

Using Hardy's inequality, we obtain

$$\begin{aligned}\left|b\big((v_n),\psi,v_n\big)\right|&\le\int_\Omega|v_n||\nabla\psi||v_n|\,dx\\&\le C_4\|\varphi\|_{L^\infty(\partial\Omega)}\int_{\operatorname{dist}(x,\partial\Omega)\le C_2'\varepsilon}\frac{|v_n|^2}{\operatorname{dist}(x,\partial\Omega)}\,dx\\&\le C_2'C_4\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\int_\Omega\frac{|v_n|^2}{[\operatorname{dist}(x,\partial\Omega)]^2}\,dx\\&\le C_2'C_3C_4\varepsilon\|\varphi\|_{L^\infty(\partial\Omega)}\|v_n\|^2,\end{aligned}\tag{3.19}$$

and choose suitable ε such that

$$\left|b\big((v_n),\psi,v_n\big)\right|\le\frac{\nu}{6}\|v_n\|^2. \tag{3.20}$$

By the Young inequality, the Hölder inequality, Hardy's inequality, the Cauchy inequality, and the property of the trilinear operator, we derive

$$\begin{aligned}\left|\langle P_n\bar{f},v_n\rangle\right|&\le\left|\langle\bar{f},v_n\rangle\right|\le\left|\langle f_{n\rho},v_n\rangle\right|+\nu\left|\langle F,v_n\rangle\right|\\&\le|f_{n\rho}||v_n|+\frac{C\nu}{\varepsilon^{\frac{3}{2}}}\|\varphi\|_{L^2(\partial\Omega)}\|v_n\|\\&\le\frac{\nu}{6}\|v_n\|^2+\frac{3}{2\nu\lambda_1}|f_{n\rho}|^2+\frac{C\nu}{\varepsilon^{\frac{3}{2}}}\|\varphi\|_{L^2(\partial\Omega)}\|v_n\|\\&\le\frac{\nu}{6}\|v_n\|^2+\frac{3}{2\nu\lambda_1}\left(a\left|v_n\big(t-\rho(t)\big)+\psi\right|^2+b\right)+\frac{C\nu}{\varepsilon^{\frac{3}{2}}}\|\varphi\|_{L^2(\partial\Omega)}\|v_n\|\\&\le\frac{\nu}{6}\|v_n\|^2+\frac{3a}{\nu\lambda_1}\left|v_n\big(t-\rho(t)\big)\right|^2+\frac{3a}{\nu\lambda_1}|\psi|^2+\frac{3b}{2\nu\lambda_1}\\&\quad+\frac{C\nu}{\varepsilon^{\frac{3}{2}}}\|\varphi\|_{L^2(\partial\Omega)}\|v_n\|\\&\le\frac{\nu}{6}\|v_n\|^2+\frac{3a}{\nu\lambda_1}\left|v_n\big(t-\rho(t)\big)\right|^2+\frac{3aC_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}+\frac{3b}{2\nu\lambda_1}\\&\quad+\frac{C\nu}{\varepsilon^{\frac{3}{2}}}\|\varphi\|_{L^2(\partial\Omega)}\|v_n\|,\end{aligned}\tag{3.21}$$

$$\begin{aligned}\left|b(\psi,\psi,v_n)\right|&\le\int_\Omega|\psi||\nabla\psi||v_n|\,dx\le C_4\|\varphi\|_{L^\infty(\partial\Omega)}\int_\Omega\frac{|v_n|}{\operatorname{dist}(x,\partial\Omega)}|\psi|\,dx\\&\le C_4\|\varphi\|_{L^\infty(\partial\Omega)}\left\{\int_\Omega\frac{|v_n|^2}{[\operatorname{dist}(x,\partial\Omega)]^2}\,dx\right\}^{1/2}\left\{\int_{\operatorname{dist}(x,\partial\Omega)\le C_2'\varepsilon}|\psi|^2\,dx\right\}^{1/2}\\&\le C\varepsilon\|\varphi\|^2_{L^\infty(\partial\Omega)}|\partial\Omega|^{1/2}\|v_n\|\sqrt{\varepsilon},\end{aligned}\tag{3.22}$$

$$\begin{aligned}\left|\langle g_n, v_n\rangle\right| &\le |g_n||v_n| \\ &\le \frac{|g_n|^2}{2C_g} + \frac{C_g}{2}|v_n|^2 \\ &\le \frac{|g_n|^2}{2C_g} + \frac{C_g\lambda_1^{-1}}{2}\|v_n\|^2. \end{aligned} \tag{3.23}$$

Combining (3.19)-(3.23), we conclude

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(|v_n|^2 + \alpha^2\|v_n\|^2\right) + \nu\|v_n\|^2 \\ &\quad\le \frac{\nu}{3}\|v_n\|^2 + \frac{3a}{\nu\lambda_1}\left|v_n\left(t-\rho(t)\right)\right|^2 + \frac{3aC_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{3b}{2\nu\lambda_1} + \frac{|g_n|^2}{2C_g} + \frac{C_g}{2\lambda_1}\|v_n\|^2 \\ &\qquad + \left(\frac{C\nu}{\varepsilon^{\frac{3}{2}}}\|\varphi\|_{L^2(\partial\Omega)} + C\varepsilon^{3/2}\|\varphi\|^2_{L^\infty(\partial\Omega)}|\partial\Omega|^{1/2}\right)\|v_n\| \\ &\quad\le \frac{\nu}{3}\|v_n\|^2 + \frac{3a}{\nu\lambda_1}\left|v_n\left(t-\rho(t)\right)\right|^2 + \frac{3aC_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{3b}{2\nu\lambda_1} + \frac{|g_n|^2}{2C_g} + \frac{C_g}{2\lambda_1}\|v_n\|^2 \\ &\qquad + \frac{\nu}{6}\|v_n\|^2 + \frac{3}{2\nu}\left(\frac{C\nu}{\varepsilon^{\frac{3}{2}}}\|\varphi\|_{L^2(\partial\Omega)} + C\varepsilon^{3/2}\|\varphi\|^2_{L^\infty(\partial\Omega)}|\partial\Omega|^{1/2}\right)^2, \end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{d}{dt}\left(|v_n|^2 + \alpha^2\|v_n\|^2\right) \\ &\quad\le \frac{6a}{\nu\lambda_1}\left|v_n\left(t-\rho(t)\right)\right|^2 + \frac{1}{C_g}|g_n|^2 + K_0^2 - \left(\nu - \frac{C_g}{\lambda_1}\right)\|v_n\|^2, \end{aligned} \tag{3.24}$$

where

$$K_0^2 = \frac{6aC_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{3b}{\nu\lambda_1} + \frac{3}{\nu}\left(\frac{C\nu}{\varepsilon^{\frac{3}{2}}}\|\varphi\|_{L^2(\partial\Omega)} + C\varepsilon^{3/2}\|\varphi\|^2_{L^\infty(\partial\Omega)}|\partial\Omega|^{1/2}\right)^2.$$

Choosing suitable $m > 0$ such that $\nu > \frac{3C_g}{\lambda_1} + \frac{m}{\lambda_1} + m\alpha^2 + \frac{6ae^{mh}}{\nu\lambda_1^2(1-M)}$, we have

$$\begin{aligned}&\frac{d}{dt}\left[e^{mt}\left(|v_n|^2 + \alpha^2\|v_n\|^2\right)\right] \\ &\quad= me^{mt}\left(|v_n|^2 + \alpha^2\|v_n\|^2\right) + e^{mt}\frac{d}{dt}\left(|v_n|^2 + \alpha^2\|v_n\|^2\right) \\ &\quad\le me^{mt}\left(|v_n|^2 + \alpha^2\|v_n\|^2\right) + e^{mt}\frac{6a}{\nu\lambda_1}\left|v_n\left(t-\rho(t)\right)\right|^2 + \frac{1}{C_g}|g_n|^2 + K_0^2 \\ &\qquad - \left(\nu - \frac{C_g}{\lambda_1}\|v_n\|^2\right) \\ &\quad\le \frac{6ae^{mt}}{\nu\lambda_1}\left|v_n\left(t-\rho(t)\right)\right|^2 + \frac{e^{mt}}{C_g}|g_n|^2 + K_0^2e^{mt} \\ &\qquad - e^{mt}\left(\nu - \frac{C_g}{\lambda_1} - \frac{m}{\lambda_1} - m\alpha^2\right)\|v_n\|^2. \end{aligned} \tag{3.25}$$

Integrating (3.25) over $[\tau, t]$, we derive

$$\begin{aligned}
&e^{mt}\left(|v_n|^2+\alpha^2\|v_n\|^2\right)-e^{m\tau}\left(\left|v_n(\tau)\right|^2+\alpha^2\left\|v_n(\tau)\right\|^2\right)\\
&\quad\le \frac{K_0^2}{m}e^{mt}+\frac{6a}{\nu\lambda_1}\int_\tau^t e^{ms}\left|v_n\left(s-\rho(s)\right)\right|^2 ds+\frac{1}{C_g}\int_\tau^t e^{ms}|g_n|^2 ds\\
&\qquad-\left(\nu-\frac{C_g}{\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left\|v_n(s)\right\|^2 ds\\
&\quad\le \frac{K_0^2}{m}e^{mt}+\frac{6ae^{mh}}{\nu\lambda_1(1-M)}\int_{\tau-h}^t e^{ms}\left|v_n(s)\right|^2 ds+C_g\int_{\tau-h}^t e^{ms}\left|v_n(s)+\psi\right|^2 ds\\
&\qquad-\left(\nu-\frac{C_g}{\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left\|v_n(s)\right\|^2 ds\\
&\quad\le \frac{K_0^2}{m}e^{mt}+\frac{6ae^{mh}}{\nu\lambda_1(1-M)}\left(\int_{\tau-h}^\tau e^{ms}\left|v_n(s)\right|^2 ds+\int_\tau^t e^{ms}\left|v_n(s)\right|^2 ds\right)\\
&\qquad+C_g\int_{\tau-h}^\tau e^{ms}\left|v_n(s)+\psi\right|^2 ds\\
&\qquad+C_g\int_\tau^t e^{ms}\left|v_n(s)+\psi\right|^2 ds-\left(\nu-\frac{C_g}{\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left\|v_n(s)\right\|^2 ds\\
&\quad\le \frac{K_0^2}{m}e^{mt}+\frac{6ae^{mh}}{\nu\lambda_1(1-M)}\left(\int_{\tau-h}^\tau e^{ms}|\phi_n-\psi|^2 ds+\int_\tau^t e^{ms}\left|v_n(s)\right|^2 ds\right)\\
&\qquad+C_g\int_{\tau-h}^\tau e^{ms}|\phi_n|^2 ds\\
&\qquad+2C_g\int_\tau^t e^{ms}\left(|v_n|^2+|\psi|^2\right) ds-\left(\nu-\frac{C_g}{\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left\|v_n(s)\right\|^2 ds\\
&\quad\le \frac{K_0^2}{m}e^{mt}+\frac{6ae^{mh}}{\nu\lambda_1(1-M)}\left(2\int_{\tau-h}^\tau e^{ms}\left(|\phi_n|^2+|\psi|^2\right) ds+\int_\tau^t e^{ms}\left|v_n(s)\right|^2 ds\right)\\
&\qquad+C_g\int_{\tau-h}^\tau e^{ms}|\phi_n|^2 ds\\
&\qquad+2C_g\int_\tau^t e^{ms}\left(|v_n|^2+|\psi|^2\right) ds-\left(\nu-\frac{C_g}{\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left\|v_n(s)\right\|^2 ds\\
&\quad\le \frac{K_0^2}{m}e^{mt}+\frac{12ae^{mh}e^{m\tau}}{\nu\lambda_1(1-M)}\int_{\tau-h}^\tau |\phi_n|^2 ds+\frac{12ae^{mh}e^{m\tau}}{\nu\lambda_1(1-M)}|\psi|^2 h\\
&\qquad+\frac{6ae^{mh}}{\nu\lambda_1(1-M)}\int_\tau^t e^{ms}\left|v_n(s)\right|^2 ds+C_g e^{m\tau}\int_{\tau-h}^\tau |\phi_n|^2 ds\\
&\qquad+2C_g\int_\tau^t e^{ms}|v_n|^2 ds+\frac{2C_g e^{mt}}{m}|\psi|^2-\left(\nu-\frac{C_g}{\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left\|v_n(s)\right\|^2 ds\\
&\quad\le \frac{K_0^2}{m}e^{mt}+\frac{12ae^{mh}e^{m\tau}}{\nu\lambda_1(1-M)}|\psi|^2 h+\left(\frac{12ae^{mh}}{\nu\lambda_1(1-M)}+C_g\right)e^{m\tau}\int_{\tau-h}^\tau |\phi_n|^2 ds\\
&\qquad+\frac{2C_g e^{mt}}{m}|\psi|^2-\left(\nu-\frac{3C_g}{\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2-\frac{6ae^{mh}}{\nu\lambda_1^2(1-M)}\right)\int_\tau^t e^{ms}\left\|v_n(s)\right\|^2 ds\\
&\quad\le \frac{K_0^2}{m}e^{mt}+\frac{12ae^{mh}e^{m\tau}}{\nu\lambda_1(1-M)}|\psi|^2 h+\left(\frac{12ae^{mh}}{\nu\lambda_1(1-M)}+C_g\right)e^{m\tau}\int_{\tau-h}^\tau |\phi_n|^2 ds\\
&\qquad+\frac{2C_g e^{mt}}{m}|\psi|^2,
\end{aligned}\tag{3.26}$$

which implies

$$\begin{aligned}
&\left|v_n(t)\right|^2 + \alpha^2\left\|v_n(t)\right\|^2 \\
&\quad\le \left|v_n(\tau)\right|^2 + \alpha^2\left\|v_n(\tau)\right\|^2 + \frac{12ae^{mh}}{\nu\lambda_1(1-M)}|\psi|^2 h \\
&\qquad + \left(\frac{12ae^{mh}}{\nu\lambda_1(1-M)} + C_g\right)\int_{-h}^0 |\phi_n|^2\,ds + \frac{K_0^2}{m} + \frac{2C_g}{m}|\psi|^2 \\
&\quad\le \left|v_n(\tau)\right|^2 + \alpha^2\left\|v_n(\tau)\right\|^2 + \frac{12ae^{mh}C_4^2}{\nu\lambda_1(1-M)}\|\varphi\|^2_{L^\infty(\partial\Omega)}h \\
&\qquad + \left(\frac{12ae^{mh}}{\nu\lambda_1(1-M)} + C_g\right)\|\phi_n\|^2_{L^2_H} + \frac{K_0^2}{m} + \frac{2C_gC_4^2}{m}\|\varphi\|^2_{L^\infty(\partial\Omega)} \equiv K_1^2.
\end{aligned} \tag{3.27}$$

Integrating (3.24) over $[t, t+1]$, we obtain

$$\begin{aligned}
&\left(\left|v_n(t+1)\right|^2 + \alpha^2\left\|v_n(t+1)\right\|^2\right) - \left(\left|v_n(t)\right|^2 + \alpha^2\left\|v_n(t)\right\|^2\right) \\
&\qquad + \left(\nu - C_g\lambda_1^{-1}\right)\int_t^{t+1}\|v_n\|^2\,ds \\
&\quad\le K_0^2 + \frac{1}{C_g}\int_t^{t+1}|g_n|^2\,ds + \frac{6a}{\nu\lambda_1}\int_t^{t+1}\left|v_n\big(s-\rho(s)\big)\right|^2\,ds \\
&\quad\le K_0^2 + C_g\int_{t-h}^{t+1}\left|v_n(s)+\psi\right|^2\,ds + \frac{6a}{\nu\lambda_1(1-M)}\int_{t-h}^{t+1}\left|v_n(s)\right|^2\,ds \\
&\quad\le K_0^2 + 2C_g\int_{t-h}^{t+1}\left(\left|v_n(s)\right|^2 + |\psi|^2\right)ds + \frac{6a}{\nu\lambda_1(1-M)}\int_{t-h}^{t+1}\left|v_n(s)\right|^2\,ds \\
&\quad\le K_0^2 + 2(h+1)C_g|\psi|^2 + \left(\frac{6a}{\nu\lambda_1(1-M)} + 2C_g\right)\int_{t-h}^{t+1}\left|v_n(s)\right|^2\,ds \\
&\quad\le \left(\frac{6a}{\nu\lambda_1(1-M)} + 2C_g\right)\left(\int_{t-h}^{t}\left|v_n(s)\right|^2\,ds + \int_t^{t+1}\left|v_n(s)\right|^2\,ds\right) + 2(h+1)C_g|\psi|^2 + K_0^2 \\
&\quad\le \left(\frac{6a}{\nu\lambda_1(1-M)} + 2C_g\right)\left(K_1^2h + \int_{-h}^0\left|\eta_n(s)\right|^2\,ds + \frac{1}{\lambda_1}\int_t^{t+1}\left\|v_n(s)\right\|^2\,ds\right) \\
&\qquad + K_0^2 + 2(h+1)C_g|\psi|^2 \\
&\quad\le \left(\frac{6a}{\nu\lambda_1(1-M)} + 2C_g\right) \\
&\qquad \times\left(K_1^2h + 2\int_{-h}^0\left(|\phi_n|^2 + |\psi|^2\right)ds + \frac{1}{\lambda_1}\int_t^{t+1}\left\|v_n(s)\right\|^2\,ds\right) \\
&\qquad + K_0^2 + 2(h+1)C_g|\psi|^2 \\
&\quad\le \left(\frac{6a}{\nu\lambda_1(1-M)} + 2C_g\right) \\
&\qquad \times\left(K_1^2h + 2\int_{-h}^0|\phi_n|^2\,ds + 2h|\psi|^2 + \frac{1}{\lambda_1}\int_t^{t+1}\left\|v_n(s)\right\|^2\,ds\right) \\
&\qquad + K_0^2 + 2(h+1)C_g|\psi|^2
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
&\left(\nu-\frac{3C_g}{\lambda_1}-\frac{6a}{\nu\lambda_1^2(1-M)}\right)\int_t^{t+1}\|v_n\|^2\,ds\\
&\quad\le K_1^2+\left(\frac{6a}{\nu\lambda_1(1-M)}+2C_g\right)\left(K_1^2h+2\int_{-h}^0|\phi_n|^2\,ds+2h|\psi|^2\right)+K_0^2\\
&\qquad+2(h+1)C_g|\psi|^2\\
&\quad\le K_1^2+\left(\frac{6a}{\nu\lambda_1(1-M)}+2C_g\right)\left(K_1^2h+2\|\phi_n\|_{L_H^2}^2+2hC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2\right)+K_0^2\\
&\qquad+2(h+1)C_gC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2\equiv K_2^2.
\end{aligned}\tag{3.29}$$

That is,

$$\int_t^{t+1}\|v_n\|^2\,ds\le\frac{K_2^2}{\nu-\frac{3C_g}{\lambda_1}-\frac{6a}{\nu\lambda_1^2(1-M)}}\equiv I_V^2,\tag{3.30}$$

which means $v_n(t)$ is uniformly bounded in $L^\infty(\tau,T;V)\cap L^2(\tau,T;V)$. Using the Alaoglu compact theorem, we can find a subsequence (still written as v_n without confusion) such that

$$v_n\to^* v\quad\text{in } L^\infty(\tau,T;V);\qquad v_n\to v\quad\text{in } L^2(\tau,T;V),\tag{3.31}$$

i.e., $v\in L^\infty(\tau,T;V)\cap L^2(\tau,T;V)$.

Next, we prove that $\frac{dv_n}{dt}$ is uniformly bounded in $L^2(\tau,T;V')$. Since

$$\frac{dv_n}{dt}=-\nu Av_n-\alpha^2Av_{nt}-B(v_n)-R(v_n)+P_n\bar{f}-B(\psi)+g_n,\tag{3.32}$$

and $v_n\in L^2(\tau,T;V)$, we derive that $-\nu Av_n,\alpha^2v_{nt},g_n\in L^2(\tau,T;V')$, and

$$\begin{aligned}
\|B(v_n)\|_{L^2(\tau,T;V')}^2&=\int_\tau^T\left(\sup_{\|u\|=1}\left|((v_n\cdot\nabla)v_n,u)\right|\right)^2ds\\
&\le\int_\tau^T\left(\left|(v_n\cdot\nabla)v_n\right||u|\right)^2ds\\
&\le C\int_\tau^T\left|(v_n\cdot\nabla)v_n\right|^2\|u\|^2\,ds\\
&=C\int_\tau^T\left|(v_n\cdot\nabla)v_n\right|^2\,ds\\
&\le C\int_\tau^T|v_n|^2|\nabla v_n|^2\,ds\\
&\le C\int_\tau^T|v_n|^2\|v_n\|^2\,ds\\
&\le C\|v_n\|_{L^\infty(\tau,T;H)}^2\|v_n\|_{L^2(\tau,T;V)}^2\\
&\le C\|v_n\|_{L^\infty(\tau,T;V)}^2\|v_n\|_{L^2(\tau,T;V)}^2.
\end{aligned}\tag{3.33}$$

Similarly, we have

$$\begin{aligned}\|R(v_n)\|^2_{L^2(\tau,T;V')} &\le C\|\psi\|^2_{L^\infty(\Omega)}\|v_n\|^2_{L^\infty(\tau,T;H)} + C|\psi|^2\|v_n\|^2_{L^2(\tau,T;V)} \\ &\le C\|\varphi\|^2_{L^\infty(\partial\Omega)}\big(\|v_n\|^2_{L^\infty(\tau,T;V)} + \|v_n\|^2_{L^2(\tau,T;V)}\big).\end{aligned} \tag{3.34}$$

Since $B(\psi) \in L^2(\tau, T; V')$, we conclude that $\frac{dv_n}{dt}$ is uniformly bounded in $L^2(\tau, T; V')$. By the compact embedding theorem, we also have

$$v_n \to v, \quad \text{in } L^2(\tau, T; V); \qquad v_n(\tau) = P_n v_\tau \to v(\tau) = v_\tau. \tag{3.35}$$

□

Theorem 3.2 *Let* $u_\tau, f \in H$, $\varphi \in L^\infty(\partial\Omega)$, *and* $\varphi \cdot n = 0$ *on* $\partial\Omega$. *Then* (1.1) *has a unique weak solution.*

Proof The family of stream functions ψ_ε was constructed in [8] which satisfied $\psi_\varepsilon \in C^\infty(\Omega)$. In addition, the solution v of (3.16) is obtained in Theorem 3.1. Let $u = v + \psi_\varepsilon$, it is easy to check that u is the weak solution of (1.1) which satisfies (i), (ii), and (iii).

Suppose that u_1 and u_2 are two solutions to (1.1) with stream functions ψ_1 and ψ_2, respectively. Let $v \in C_0^\infty(\Omega)$, div $v = 0$, from the condition (ii) we get

$$\begin{aligned}&\frac{d}{dt}\langle u_1 - u_2, v\rangle - \nu\langle u_1 - u_2, \Delta v\rangle - \alpha^2\frac{d}{dt}\langle u_1 - u_2, \Delta v\rangle \\ &\quad= \int_\Omega \sum_{i,j=1}^2 \big(u_1^i u_1^j - u_2^i u_2^j\big)\frac{\partial v^i}{x_j}\,dx + \big\langle f_\rho(u_1) - f_\rho(u_2), v\big\rangle \\ &\qquad+ \bigg\langle \int_{-h}^0 \big[G\big(s, u_1(t+s)\big) - G_2\big(s, u(t+s)\big)\big]\,ds, v\bigg\rangle.\end{aligned} \tag{3.36}$$

We claim that (3.36) holds for any $v \in V$. In fact, from the condition (ii), we have

$$u_1 - u_2 = (u_1 - \psi_1) - (u_2 - \psi_2) + (\psi_1 - \psi_2) \in L^2\big([0,T]; V\big), \tag{3.37}$$

thus we can write $\langle u_1 - u_2, \Delta v\rangle = -(u_1 - u_2, v)$ $(l = 1, 2)$,

$$\begin{aligned}\left(\int_\Omega |u_l|^4\,dx\right)^{1/4} &\le \left(\int_\Omega |u_l - \psi_l|^4\,dx\right)^{1/4} + \left(\int_\Omega |\psi_l|^4\,dx\right)^{1/4} \\ &\le C\left(\int_\Omega \big|\nabla(u_l - \psi_l)\big|^2\,dx\right)^{1/4} \\ &\quad\times \left(\int_\Omega |u_l - \psi_l|^2\,dx\right)^{1/4}\left(\int_\Omega |\psi_l|^4\,dx\right)^{1/4},\end{aligned} \tag{3.38}$$

$$\left|\int_\Omega \sum_{i,j=1}^2 u_l^i u_l^j \frac{\partial v^i}{x_j}\,dx\right| \le C\left(\int_\Omega |u_l|^4\,dx\right)^{1/2}\left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2}, \tag{3.39}$$

and $u_l \in L^4(\Omega \times (\tau, T))$, thus

$$\frac{d}{dt}(u_1 - u_2) \in L^2\big([\tau, T]; V'\big), \quad \text{for } v \in V, \tag{3.40}$$

which implies (3.36) holds for any $v \in V$.

Let $v = u_1 - u_2$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(|v|^2 + \alpha^2\|v\|^2\big) + \nu\|v\|^2 \\
&\quad\leq \left|\int_\Omega \sum_{i,j=1}^{2}\big(u_1^i u_1^j - u_2^i u_2^j\big)\frac{\partial v^i}{x_j}\,dx\right| + \big|\langle f_\rho(u_1) - f_\rho(u_2), v\rangle\big| \\
&\qquad + \big|\langle g(t,u_{1t}) - g(t,u_{2t}), v\rangle\big| \\
&\quad\leq \left|\int_\Omega \sum_{i,j=1}^{2}\left(u_1^i v^j\frac{\partial v^i}{x_j} + u_2^j\frac{1}{2}\frac{\partial u_2^j}{x_j}\right)dx\right| + \big|\langle f_\rho(u_1) - f_\rho(u_2), v\rangle\big| \\
&\qquad + \big|\langle g(t,u_{1t}) - g(t,u_{2t}), v\rangle\big| \\
&\quad\leq \left|\int_\Omega \sum_{i,j=1}^{2} u_1^i v^j\frac{\partial v^i}{x_j}\,dx\right| + \big|\langle f_\rho(u_1) - f_\rho(u_2), v\rangle\big| \\
&\qquad + \big|\langle g(t,u_{1t}) - g(t,u_{2t}), v\rangle\big| \\
&\quad\leq C\left(\int_\Omega |u_1|^4\,dx\right)^{1/4}\left(\int_\Omega |v|^{4/3}|\nabla v|^{4/3}\,dx\right)^{3/4} + L(\beta)\big|v\big(t-\rho(t)\big)\big||v| + L_g\|v\|_{C_H}|v| \\
&\quad\leq C\left(\int_\Omega |u_1|^4\,dx\right)^{1/4}\left(\left(\int_\Omega |v|^4\,dx\right)^{\frac{1}{3}}\left(\int_\Omega |\nabla v|^{4/3}\,dx\right)^{\frac{2}{3}}\right)^{3/4} + C\|v\|_{C_H}|v| \\
&\quad\leq C\|u_1\|_{L^4}\|v\|_{L^4}|\nabla v| + C\|v\|_{C_H}|v| \\
&\quad\leq C\|u_1\|_{L^4}\|\nabla v\|^{\frac{1}{2}}|v|^{\frac{1}{2}}|\nabla v| + C\|v\|_{C_H}|v| \\
&\quad\leq \nu\|v\|^2 + C_\nu\|u_1\|_{L^4}^4|v|^2 + C\|v\|_{C_H}|v|,
\end{aligned} \tag{3.41}$$

which means

$$\frac{d}{dt}\big(\|v\|^2\big) \leq C_\nu\|u_1\|_{L^4}^4|v|^2 + C\|v\|_{C_H}|v|.$$

Since $u_1 \in L^4(\Omega \times (\tau, T))$ and $v(\cdot,\tau) = 0$, we derive $v = 0$ which means the uniqueness of solution holds. □

3.4 Continuous dependence of initial data

Consider the two solutions $u(\cdot)$ and $v(\cdot)$ to problem (1.1) with corresponding initial data (u_τ, ϕ_1) and (v_τ, ϕ_2), respectively. Let $w = u - v$, then w satisfies the problem

$$\begin{cases}
\frac{dw}{dt} - \nu\Delta w - \alpha^2\Delta w_t + (u\cdot\nabla)u - (v\cdot\nabla)v & \\
\quad = g(u_t) - g(v_t) + f_\rho(u) - f_\rho(v), & \\
\operatorname{div} w = 0, & (x,t) \in \Omega_\tau, \\
w(t,x)|_{\partial\Omega} = 0, & (x,t) \in \partial\Omega_\tau, \\
w(\tau,x) = u_\tau(x) - v_\tau(x), & x \in \Omega, \\
u(t,x) = \phi_1(t-\tau,x) - \phi_2(t-\tau,x), & (x,t) \in \Omega_{\tau h}.
\end{cases} \tag{3.42}$$

Since $B(u,u) - B(v,v) = B(w,u) + B(v,w)$, we obtain the abstract form

$$\frac{dw}{dt} + \nu Aw + \alpha^2 Aw_t + B(w,u) + B(v,w) = g(u_t) - g(v_t) + +f_\rho(u) - f_\rho(v). \tag{3.43}$$

Multiplying (3.43) by w, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigl(|w|^2 + \alpha^2\|w\|^2\bigr) + \nu\|w\|^2 \\
&\quad\le \bigl|b(w,u,w)\bigr| + \bigl|f_\rho(u) - f_\rho(v)\bigr||w| + \bigl|g(u_t) - g(v_t)\bigr||w| \\
&\quad\le C|w|\|w\|\|u\| + L(\beta)|w|\bigl|w\bigl(t-\rho(t)\bigr)\bigr| + \bigl|g(u_t) - g(v_t)\bigr||w| \\
&\quad\le \frac{\nu}{4}\|w\|^2 + \frac{C^2}{\nu}|w|^2\|u\|^2 + \frac{\nu}{8}\|w\|^2 + \frac{2L^2(\beta)}{\nu\lambda_1}\bigl|w\bigl(t-\rho(t)\bigr)\bigr|^2 + \frac{\nu}{8}\|w\|^2 \\
&\qquad + \frac{2}{\nu\lambda_1}\bigl|g(t,u_t) - g(t,v_t)\bigr|^2 \\
&\quad= \frac{\nu}{2}\|w\|^2 + \frac{C^2}{\nu}|w|^2\|u\|^2 + \frac{2L^2(\beta)}{\nu\lambda_1}\bigl|w\bigl(t-\rho(t)\bigr)\bigr|^2 \\
&\qquad + \frac{2}{\nu\lambda_1}\bigl|g(t,u_t) - g(t,v_t)\bigr|^2,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{d}{dt}\bigl(|w|^2 + \alpha^2\|w\|^2\bigr) + \nu\|w\|^2 \\
&\quad\le \frac{2C^2}{\nu}|w|^2\|u\|^2 + \frac{4L^2(\beta)}{\nu\lambda_1}\bigl|w\bigl(t-\rho(t)\bigr)\bigr|^2 + \frac{4}{\nu\lambda_1}\bigl|g(t,u_t) - g(t,v_t)\bigr|^2.
\end{aligned} \tag{3.44}$$

Noting

$$\begin{aligned}
&\int_\tau^t \bigl|w\bigl(s-\rho(s)\bigr)\bigr|^2\,ds \le \frac{1}{1-M}\int_{\tau-h}^t \bigl|w(s)\bigr|^2\,ds, \\
&\int_\tau^t \bigl|g(s,u_s) - g(s,v_s)\bigr|^2\,ds \le C_g^2\int_{\tau-h}^t \bigl|u(s) - v(s)\bigr|^2\,ds = C_g^2\int_{\tau-h}^t \bigl|w(s)\bigr|^2\,ds,
\end{aligned}$$

and integrating (3.44) over $[\tau,t]$, we obtain

$$\begin{aligned}
&|w|^2 + \alpha^2\|w\|^2 + \nu\int_\tau^t \|w\|^2\,ds \\
&\quad\le \bigl|w(\tau)\bigr|^2 + \alpha^2\bigl\|w(\tau)\bigr\|^2 + \frac{2C^2}{\nu}\int_\tau^t |w|^2\|u\|^2\,ds + \frac{4L^2(\beta)}{\nu\lambda_1}\int_\tau^t \bigl|w\bigl(s-\rho(s)\bigr)\bigr|^2\,ds \\
&\qquad + \frac{4}{\nu\lambda_1}\int_\tau^t \bigl|g(t,u_s) - g(t,v_s)\bigr|^2\,ds \\
&\quad\le \bigl|w(\tau)\bigr|^2 + \alpha^2\bigl\|w(\tau)\bigr\|^2 + \frac{2C^2}{\nu}\int_\tau^t |w|^2\|u\|^2\,ds + \frac{4L^2(\beta)}{\nu\lambda_1(1-M)}\int_{\tau-h}^t \bigl|w(s)\bigr|^2\,ds \\
&\qquad + \frac{4C_g^2}{\nu\lambda_1}\int_{\tau-h}^t \bigl|w(s)\bigr|^2\,ds \\
&\quad\le \bigl|w(\tau)\bigr|^2 + \alpha^2\bigl\|w(\tau)\bigr\|^2 + \left(\frac{4L^2(\beta)}{\nu\lambda_1(1-M)} + \frac{4C_g^2}{\nu\lambda_1}\right)\int_{\tau-h}^\tau \bigl|w(s)\bigr|^2\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_\tau^t \left(\frac{2C^2}{\nu}\|u\|^2+\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1}\right)\left|w(s)\right|^2 ds \\
&\leq \left|w(\tau)\right|^2+\alpha^2\left\|w(\tau)\right\|^2+\left(\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1}\right)\|\phi_1-\phi_2\|^2_{L^2_H} \\
&+\int_\tau^t \left(\frac{2C^2}{\nu}\|u\|^2+\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1}\right)\left|w(s)\right|^2 ds.
\end{aligned}
\tag{3.45}
$$

Since $u(t)\in L^\infty(\tau,T;V)\cap L^2(\tau,T;V)$, neglecting the integrating term on left side of (3.45), putting $s\in(t-h,t)$ instead of t and using the Gronwall inequality to (3.45), we see

$$
\begin{aligned}
\left|w(s)\right|^2 \leq{}& \left(\left|w(\tau)\right|^2+\alpha^2\left\|w(\tau)\right\|^2+\left(\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1}\right)\|\phi_1-\phi_2\|^2_{L^2_H}\right) \\
&\times e^{\int_\tau^t(\frac{2C^2}{\nu}\|u\|^2+\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1})\,ds}.
\end{aligned}
$$

Similarly, using the Poincaré inequality, we get

$$
\begin{aligned}
\left\|w(s)\right\|^2 \leq{}& \frac{1}{\alpha^2}\left(\left|w(\tau)\right|^2+\alpha^2\left\|w(\tau)\right\|^2+\left(\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1}\right)\|\phi_1-\phi_2\|^2_{L^2_H}\right) \\
&\times e^{\frac{1}{\lambda_1}\int_\tau^t(\frac{2C^2}{\nu}\|u\|^2+\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1})\,ds}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\int_\tau^t \left\|w(s)\right\|^2 ds \\
&\leq \frac{1}{\nu\alpha^2}\left(\left|w(\tau)\right|^2+\alpha^2\left\|w(\tau)\right\|^2+\left(\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1}\right)\|\phi_1-\phi_2\|^2_{L^2_H}\right) \\
&\times\left(1+e^{\int_\tau^t(\frac{2C^2}{\nu}\|u\|^2+\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1})\,ds}\int_\tau^t\left(\frac{2C^2}{\nu}\|u\|^2+\frac{4L^2(\beta)}{\nu\lambda_1(1-M)}+\frac{4C_g^2}{\nu\lambda_1}\right)ds\right).
\end{aligned}
$$

This implies the continuous dependence on the initial data for the solution which generates a continuous process $\{\tilde{U}(\cdot,\cdot)\}$.

4 Existence of absorbing sets

In this section we shall derive the existence of pullback absorbing sets for the 2D Navier-Stokes-Voigt equations with continuous delay and distributed delay on the Lipschitz domain.

From Theorem 3.1, we obtain the process $\tilde{U}(\cdot,\tau;(v_\tau,\phi))=v_t(\cdot;(v_\tau,\phi))$, where $(v_\tau,\phi)\in V\times L^2_H$, $V\times L^2_H$ is a Hilbert space, and the corresponding norm can be defined as

$$
\left\|(v_\tau,\phi)\right\|^2_{V\times L^2_H}=|v_\tau|^2+\alpha^2\|v_\tau\|^2+\|\phi\|^2_{L^2_H},\quad (v_\tau,\phi)\in V\times L^2_H.
$$

To derive the existence of pullback attractor, we need to prove the existence of pullback absorbing set of $\tilde{U}(\cdot,\tau;(v_\tau,\phi))$ in C_V first of all thus:

Theorem 4.1 *Let* $(v_\tau,\eta)\in V\times L^2_H$, *and the assumptions* (a)-(h) *hold, then there exists a pullback absorbing set in* C_V *for the system* (3.15).

Proof Let $D \subset V \times L^2_H$ be any bounded set, and $(v_\tau, \phi) \in D$, then there exists a constant $d > 0$ such that

$$|v_\tau|^2 + \alpha^2 \|v_\tau\|^2 + \|\phi\|^2_{L^2_H} \le d^2. \tag{4.1}$$

Similar to the proof of Theorem 3.1, we get

$$\frac{d}{dt}\left(|v|^2 + \alpha^2\|v\|^2\right) \le \frac{6a}{\nu\lambda_1}\left|v\left(t-\rho(t)\right)\right|^2 + \frac{1}{C_g}|g|^2 + K_0^2 - \left(\nu - \frac{C_g}{\lambda_1}\right)\|v_n\|^2. \tag{4.2}$$

Choose a suitable constant $m > 0$ such that $\nu > \frac{C_g}{\lambda_1} + \frac{m}{\lambda_1} + m\alpha^2 + \frac{6ae^{mh}}{\nu\lambda_1^2(1-M)}$ and

$$\begin{aligned}\frac{d}{dt}\left(e^{mt}\left(|v|^2 + \alpha^2\|v\|^2\right)\right) &\le \frac{6ae^{mt}}{\nu\lambda_1}\left|v\left(t-\rho(t)\right)\right|^2 + \frac{e^{mt}}{C_g}|g|^2 + K_0^2 e^{mt} \\ &\quad - e^{mt}\left(\nu - \frac{C_g}{\lambda_1} - \frac{m}{\lambda_1} - m\alpha^2\right)\|v_n\|^2. \end{aligned}\tag{4.3}$$

Integrating (4.3) over $[\tau, t]$, we have

$$\begin{aligned}&e^{mt}\left(\left|v(t)\right|^2 + \alpha^2\left\|v(t)\right\|^2\right) - e^{m\tau}\left(\left|v(\tau)\right|^2 + \alpha^2\left\|v(\tau)\right\|^2\right) \\ &\quad\le K_0^2\int_\tau^t e^{ms}\,ds + \frac{6a}{\nu\lambda_1}\int_\tau^t e^{ms}\left|v\left(s-\rho(s)\right)\right|^2 + \frac{1}{C_g}\int_\tau^t e^{ms}|g|^2\,ds \\ &\qquad - \left(\nu - \frac{3C_g}{\lambda_1} - \frac{m}{\lambda_1} - m\alpha^2\right)\int_\tau^t e^{ms}\left\|v(s)\right\|^2\,ds \\ &\quad\le \frac{K_0^2 e^{mt}}{m} + e^{m\tau}\left(\frac{12ae^{mh}}{\nu\lambda_1(1-M)}|\psi|^2 h + \left(\frac{12ae^{mh}}{\nu\lambda_1(1-M)} + C_g\right)\int_{-h}^0|\phi|^2\,ds\right) \\ &\qquad + \frac{2C_g}{m}|\psi|^2 e^{mt} - \left(\nu - \frac{C_g}{\lambda_1} - \frac{m}{\lambda_1} - m\alpha^2 - \frac{6ae^{mh}}{\nu\lambda_1^2(1-M)}\right)\int_\tau^t e^{ms}\left\|v(s)\right\|^2\,ds \\ &\quad\le \frac{K_0^2 e^{mt}}{m} + e^{m\tau}\left(\frac{12ae^{mh}}{\nu\lambda_1(1-M)}|\psi|^2 h + \left(\frac{12ae^{mh}}{\nu\lambda_1(1-M)} + C_g\right)\|\phi\|^2_{L^2_H}\right) + \frac{2C_g}{m}|\psi|^2 e^{mt},\end{aligned}$$

which implies

$$\begin{aligned}&\left|v(t)\right|^2 + \alpha^2\left\|v(t)\right\|^2 \\ &\quad\le e^{m(\tau-t)}\Bigg(\left|v(\tau)\right|^2 + \alpha^2\left\|v(\tau)\right\|^2 \\ &\qquad + \frac{12ae^{mh}}{\nu\lambda_1(1-M)}|\psi|^2 h + \left(\frac{12ae^{mh}}{\nu\lambda_1(1-M)} + C_g\right)\|\phi\|^2_{L^2_H}\Bigg) \\ &\qquad + \frac{K_0^2}{m} + \frac{2C_g}{m}|\psi|^2 \\ &\quad\le e^{m(\tau-t)}\left(1 + \frac{12ae^{mh}}{\nu\lambda_1(1-M)} + C_g\right)d^2 + \frac{K_0^2}{m} \\ &\qquad + \left(\frac{2C_g}{m} + \frac{12ae^{mh}}{\nu\lambda_1(1-M)}h\right)C_4^2\|\varphi\|^2_{L^\infty(\partial\Omega)}.\end{aligned}\tag{4.4}$$

Choosing $\sigma \in [-h, 0]$, and substituting t with $t+\sigma$, we have

$$\begin{aligned}
&\left|v(t+\sigma)\right|^2 + \alpha^2\left\|v(t+\sigma)\right\|^2 \\
&\quad \le e^{m(\tau-t-\sigma)}\left(1 + \frac{12ae^{mh}}{\nu\lambda_1(1-M)} + C_g\right)d^2 + \frac{K_0^2}{m} \\
&\qquad + \left(\frac{2C_g}{m} + \frac{12ae^{mh}}{\nu\lambda_1(1-M)}h\right)C_4^2\|\varphi\|^2_{L^\infty(\partial\Omega)}. \qquad (4.5)
\end{aligned}$$

Denoting $v_t(\cdot;(v_\tau,\phi))$ as $\tilde{U}(\cdot;\tau,(v_\tau,\phi))$, we have

$$\begin{aligned}
\|v_t\|^2 &= \left\|\tilde{U}\left(\cdot;\tau,(v_\tau,\phi)\right)\right\|^2_{C_V} \\
&\le e^{mh}e^{m(\tau-t)}\left(1 + \frac{12ae^{mh}}{\nu\lambda_1(1-M)} + C_g\right)d^2 + \frac{K_0^2}{m} \\
&\quad + \left(\frac{2C_g}{m} + \frac{12ae^{mh}}{\nu\lambda_1(1-M)}h\right)C_4^2\|\varphi\|^2_{L^\infty(\partial\Omega)}. \qquad (4.6)
\end{aligned}$$

Let $\rho_V^2 = \frac{2K_0^2}{m} + (\frac{4C_g}{m} + \frac{24ae^{mh}}{\nu\lambda_1(1-M)}h)C_4^2\|\varphi\|^2_{L^\infty(\partial\Omega)}$, for any $(v_\tau,\phi) \in V \times L_H^2$, when

$$\tau \le T_V = \frac{1}{m}\ln\frac{\frac{K_0^2}{m} + (\frac{2C_g}{m} + \frac{12ae^{mh}}{\nu\lambda_1(1-M)}h)C_4^2\|\varphi\|^2_{L^\infty(\partial\Omega)}}{(1 + \frac{12ae^{mh}}{\nu\lambda_1(1-M)} + C_g)d^2} - h + t,$$

$\tilde{U}(\cdot,\cdot)D \subset B_V(0,\rho_V)$, where $B_V(0,\rho_V)$ is a pullback absorbing ball centered at 0 with radius ρ_V in C_V, which completes the proof. □

Theorem 4.2 *Assume that the assumptions* (a)-(h) *hold, and* $(v_\tau,\eta) \in D(A^{\frac{3}{4}}) \times L_H^2$, *then there exists a pullback absorbing set in* $C([\tau,T],D(A^{\frac{3}{4}}))$ *for the system* (3.15).

Proof Let $D \subset D(A^{3/4}) \times L_H^2$ be any bounded set, and $(v_\tau,\phi) \in D$, then there exists a constant $d > 0$ such that

$$\left|A^{1/4}v(\tau)\right|^2 + \alpha^2\left|A^{3/4}v(\tau)\right|^2 + \|\phi\|^2_{L_H^2} \le d^2.$$

Multiplying (3.15) by $A^{\frac{1}{2}}v$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left|A^{1/4}v\right|^2 + \alpha^2\left|A^{3/4}v\right|^2\right) + \nu\left|A^{3/4}v\right|^2 \\
&\quad \le \left|b\left(v,v,A^{\frac{1}{2}}v\right)\right| + \left|b\left(v,\psi,A^{\frac{1}{2}}v\right)\right| + \left|b\left(\psi,v,A^{\frac{1}{2}}v\right)\right| + \left|b\left(\psi,\psi,A^{\frac{1}{2}}v\right)\right| \\
&\qquad + \left|\left\langle f+\nu F,A^{\frac{1}{2}}v\right\rangle\right| + \left|\left\langle g,A^{\frac{1}{2}}v\right\rangle\right|. \qquad (4.7)
\end{aligned}$$

Next, we shall estimate term by term the right side of (4.7).

$$\begin{aligned}
\left|b\left(v,v,A^{\frac{1}{2}}v\right)\right| &\le \int_\Omega |v||\nabla v|\left|A^{\frac{1}{2}}v\right|dx \le \|v\|_4|\nabla v|\left\|A^{\frac{1}{2}}v\right\|_4 \\
&\le C_1^2\left|A^{\frac{1}{4}}v\right|\left|A^{\frac{1}{2}}v\right|\left|A^{\frac{3}{4}}v\right| \le \frac{\nu}{12}\left|A^{\frac{3}{4}}v\right|^2 + \frac{3C_1^4}{\nu}\left|A^{\frac{1}{4}}v\right|^2\left|A^{\frac{1}{2}}v\right|^2, \qquad (4.8)
\end{aligned}$$

$$\begin{aligned}
\left|b\left(v,\psi,A^{\frac{1}{2}}v\right)\right| &\leq \int_\Omega |v||\nabla\psi|\left|A^{\frac{1}{2}}v\right|dx \\
&= \int_\Omega \frac{|v|}{\operatorname{dist}(x,\partial\Omega)}|\nabla\psi|\operatorname{dist}(x,\partial\Omega)\left|A^{\frac{1}{2}}v\right|dx \\
&\leq C_4\|\varphi\|_{L^\infty(\partial\Omega)}\int_\Omega \frac{|v|}{\operatorname{dist}(x,\partial\Omega)}\left|A^{\frac{1}{2}}v\right|dx \\
&\leq C_4\|\varphi\|_{L^\infty(\partial\Omega)}\left(\int_\Omega \frac{|v|^2}{[\operatorname{dist}(x,\partial\Omega)]^2\,dx}\right)^{1/2}\left(\int_\Omega\left|A^{\frac{1}{2}}v\right|^2dx\right)^{1/2} \\
&\leq C_3^{\frac{1}{2}}C_4\|\varphi\|_{L^\infty(\partial\Omega)}\left(\int_\Omega|\nabla v|^2\,dx\right)^{1/2}\left(\int_\Omega\left|A^{\frac{1}{2}}v\right|^2dx\right)^{1/2} \\
&\leq C_3^{\frac{1}{2}}C_4\|\varphi\|_{L^\infty(\partial\Omega)}\left|A^{\frac{1}{2}}v\right|^2 \leq C_2C_3^{\frac{1}{2}}C_4\|\varphi\|_{L^\infty(\partial\Omega)}\left|A^{\frac{1}{4}}v\right|\left|A^{\frac{3}{4}}v\right| \\
&\leq \frac{\nu}{12}\left|A^{\frac{3}{4}}v\right|^2 + \frac{3C_2^2C_3C_4^2}{\nu}\|\varphi\|^2_{L^\infty(\partial\Omega)}\left|A^{\frac{1}{4}}v\right|^2, \qquad (4.9)\\
\left|b\left(\psi,v,A^{\frac{1}{2}}v\right)\right| &\leq \int_\Omega|\psi||\nabla v|\left|A^{\frac{1}{2}}v\right|dx \\
&\leq C_4\|\varphi\|_{L^\infty(\partial\Omega)}\left|A^{\frac{1}{2}}v\right|^2 \\
&\leq C_2C_4\|\varphi\|_{L^\infty(\partial\Omega)}\left|A^{\frac{1}{4}}v\right|\left|A^{\frac{3}{4}}v\right| \\
&\leq \frac{\nu}{12}\left|A^{\frac{3}{4}}v\right|^2 + \frac{3C_2^2C_4^2}{\nu}\|\varphi\|^2_{L^\infty(\partial\Omega)}\left|A^{\frac{1}{4}}v\right|^2, \qquad (4.10)\\
\left|b\left(\psi,\psi,A^{\frac{1}{2}}v\right)\right| &\leq \int_\Omega|\psi||\nabla\psi|\left|A^{\frac{1}{2}}v\right|dx \\
&\leq C_4\|\varphi\|_{L^\infty(\partial\Omega)}\int_\Omega\frac{|A^{\frac{1}{2}}v|}{[\operatorname{dist}(x,\partial\Omega)]^{1/2}}|\nabla\psi|\left[\operatorname{dist}(x,\partial\Omega)\right]^{1/2}dx \\
&\leq C_4\|\varphi\|_{L^\infty(\partial\Omega)}\left(\int_\Omega\frac{|A^{\frac{1}{2}}v|^2}{[\operatorname{dist}(x,\partial\Omega)]}\,dx\right)^{1/2} \\
&\quad\times\left(\int_\Omega|\nabla\psi|^2\left[\operatorname{dist}(x,\partial\Omega)\right]dx\right)^{1/2} \\
&\leq \frac{\nu}{12}\left|A^{\frac{3}{4}}v\right|^2 + \frac{3C_0^2C_4^2C_5^2}{\nu}\|\varphi\|^2_{L^\infty(\partial\Omega)}\|\varphi\|^2_{L^2(\partial\Omega)}, \qquad (4.11)
\end{aligned}$$

$$\begin{aligned}
&\left|\left\langle f+\nu F,A^{\frac{1}{2}}v\right\rangle\right| \\
&\quad\leq \left|\left\langle f,A^{\frac{1}{2}}v\right\rangle\right| + \nu\left|\left\langle F,A^{\frac{1}{2}}v\right\rangle\right| \leq |f|\left|A^{\frac{1}{2}}v\right| + \nu\int_\Omega|F|\left|A^{\frac{1}{2}}v\right|dx \\
&\quad\leq |f|\left|A^{\frac{1}{2}}v\right| + C_2'\nu\sqrt{\varepsilon}\int_\Omega|F|\frac{|A^{\frac{1}{2}}v|}{[\operatorname{dist}(x,\partial\Omega)]^{1/2}}\,dx \\
&\quad\leq |f|\left|A^{\frac{1}{2}}v\right| + C_2'\nu\sqrt{\varepsilon}|F|\left(\int_\Omega\frac{|A^{\frac{1}{2}}v|^2}{[\operatorname{dist}(x,\partial\Omega)]}\,dx\right)^{1/2} \\
&\quad\leq |f|\left|A^{\frac{1}{2}}v\right| + C_2'C_0^{\frac{1}{2}}\nu\sqrt{\varepsilon}|F|\left|A^{\frac{3}{4}}v\right| \leq \frac{|f|}{\lambda_1^{1/4}}\left|A^{\frac{3}{4}}v\right| + C_2'C_0^{\frac{1}{2}}\nu\sqrt{\varepsilon}|F|\left|A^{\frac{3}{4}}v\right| \\
&\quad\leq \frac{\nu}{24}\left|A^{\frac{3}{4}}v\right|^2 + \frac{6|f|^2}{\nu\lambda_1^{1/2}} + \frac{\nu}{24}\left|A^{\frac{3}{4}}v\right|^2 + \frac{6C_2'^2C_0\varepsilon|F|^2}{\nu}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\nu}{12}\left|A^{\frac{3}{4}}v\right|^2 + \frac{6}{\nu\lambda_1^{1/2}}\left(a\left|v\left(t-\rho(t)\right)+\psi\right|^2+b\right) + \frac{6C_2'^2C_0\varepsilon}{\nu}\frac{C^2}{\varepsilon^3}\|\varphi\|^2_{L^2(\partial\Omega)} \\
&\le \frac{\nu}{12}\left|A^{\frac{3}{4}}v\right|^2 + \frac{12a}{\nu\lambda_1^{1/2}}\left|v\left(t-\rho(t)\right)\right|^2 + \frac{12}{\nu\lambda_1^{1/2}}|\psi|^2 + \frac{6b}{\nu\lambda_1^{1/2}} \\
&\quad + \frac{6C_2'^2C_0C^2}{\nu\varepsilon^2}\|\varphi\|^2_{L^2(\partial\Omega)} \\
&\le \frac{\nu}{12}\left|A^{\frac{3}{4}}v\right|^2 + \frac{12a}{\nu\lambda_1^{1/2}}\left|v\left(t-\rho(t)\right)\right|^2 + \frac{12C_4^2}{\nu\lambda_1^{1/2}}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{6b}{\nu\lambda_1^{1/2}} \\
&\quad + \frac{6C_2'^2C_0C^2}{\nu\varepsilon^2}\|\varphi\|^2_{L^2(\partial\Omega)},
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
\left|\langle g, A^{\frac{1}{2}}v\rangle\right| &\le |g|\left|A^{\frac{1}{2}}v\right| \le C_2|g|\left|A^{\frac{1}{4}}v\right|^{1/2}\left|A^{\frac{3}{4}}v\right|^{1/2} \\
&\le \frac{\nu}{12}\left|A^{\frac{3}{4}}v\right|^2 + \frac{C_2^2C_g}{3\nu}\left|A^{\frac{1}{4}}v\right|^2 + \frac{|g|^2}{2C_g}.
\end{aligned} \tag{4.13}
$$

Combining (4.7)-(4.13), we conclude

$$
\begin{aligned}
&\frac{d}{dt}\left(\left|A^{1/4}v\right|^2 + \alpha^2\left|A^{3/4}v\right|^2\right) \\
&\quad \le \frac{24a}{\nu\lambda_1^{1/2}}\left|v\left(t-\rho(t)\right)\right|^2 + \frac{|g|^2}{C_g} + K_3^2 \\
&\quad\quad - \left(\nu - \frac{6C_1^4}{\nu\lambda_1}K_1^2 - \frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right. \\
&\quad\quad \left. - \frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} - \frac{2C_2C_g}{3\nu\lambda_1}\right)\left|A^{3/4}v\right|^2,
\end{aligned} \tag{4.14}
$$

where

$$
K_3^2 = \frac{6C_0^2C_4^2C_5^2}{\nu}\|\varphi\|^2_{L^\infty(\partial\Omega)}\|\varphi\|^2_{L^2(\partial\Omega)} + \frac{24C_4^2}{\nu\lambda_1^{\frac{1}{2}}}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{12b}{\nu\lambda_1^{\frac{1}{2}}} + \frac{12C_2'^2C_0C^2}{\nu\varepsilon^2}\|\varphi\|^2_{L^2(\partial\Omega)}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\left[e^{mt}\left(\left|A^{1/4}v\right|^2 + \alpha^2\left|A^{3/4}v\right|^2\right)\right] \\
&\quad \le \frac{24ae^{mt}}{\nu\lambda_1^{1/2}}\left|v\left(t-\rho(t)\right)\right|^2 + \frac{e^{mt}}{C_g}|g|^2 + e^{mt}K_3^2 \\
&\quad\quad - e^{mt}\left(\nu - \frac{6C_1^4}{\nu\lambda_1}K_1^2 - \frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right. \\
&\quad\quad \left. - \frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} - \frac{2C_2C_g}{3\nu\lambda_1} - \frac{m}{\lambda_1} - m\alpha^2\right)\left|A^{3/4}v\right|^2.
\end{aligned} \tag{4.15}
$$

Choosing a suitable $m > 0$ such that

$$
\begin{aligned}
\nu &> \frac{6C_1^4}{\nu\lambda_1}K_1^2 + \frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)} + \frac{2C_2C_g}{3\nu\lambda_1} + \frac{m}{\lambda_1} + m\alpha^2 \\
&\quad + \frac{24ae^{mh}}{\nu\lambda_1^2(1-M)} + \frac{2C_g}{\lambda_1^{\frac{3}{2}}},
\end{aligned}
$$

and integrating (4.15) over $[\tau,t]$, we obtain

$$
\begin{aligned}
&e^{mt}\left(\left|A^{1/4}v\right|^2+\alpha^2\left|A^{3/4}v\right|^2\right)-e^{m\tau}\left(\left|A^{1/4}v(\tau)\right|^2+\alpha^2\left|A^{3/4}v(\tau)\right|^2\right)\\
&\quad\le\frac{e^{mt}}{m}K_3^2+\frac{24a}{\nu\lambda_1^{1/2}}\int_\tau^t e^{ms}\left|v\left(s-\rho(s)\right)\right|^2ds+\frac{1}{C_g}\int_\tau^t e^{ms}|g|^2\,ds\\
&\qquad-\left(\nu-\frac{6C_1^4}{\nu\lambda_1}K_1^2-\frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}-\frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right.\\
&\qquad\left.-\frac{2C_2C_g}{3\nu\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left|A^{3/4}v\right|^2ds\\
&\quad\le\frac{e^{mt}}{m}K_3^2+\frac{24ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}\int_{\tau-h}^t e^{ms}\left|v(s)\right|^2ds+C_g\int_{\tau-h}^t e^{ms}\left|v(s)+\psi\right|^2ds\\
&\qquad-\left(\nu-\frac{6C_1^4}{\nu\lambda_1}K_1^2-\frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}-\frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right.\\
&\qquad\left.-\frac{2C_2C_g}{3\nu\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left|A^{3/4}v\right|^2ds\\
&\quad\le\frac{24ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}\left(\int_{\tau-h}^\tau e^{ms}|\phi-\psi|^2\,ds+\int_\tau^t e^{ms}\left|v(s)\right|^2ds\right)\\
&\qquad+C_g\left(\int_{\tau-h}^\tau e^{ms}|\phi|^2\,ds+2\int_\tau^t e^{ms}\left(|v|^2+|\psi|^2\right)ds\right)+\frac{e^{mt}}{m}K_3^2\\
&\qquad-\left(\nu-\frac{6C_1^4}{\nu\lambda_1}K_1^2-\frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right.\\
&\qquad\left.-\frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}-\frac{2C_2C_g}{3\nu\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left|A^{3/4}v\right|^2ds\\
&\quad\le\frac{24ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}\left(2e^{m\tau}\int_{\tau-h}^\tau\left(|\phi|^2+|\psi|^2\right)ds+\int_\tau^t e^{ms}\left|v(s)\right|^2ds\right)\\
&\qquad+C_g\left(e^{m\tau}\int_{\tau-h}^\tau|\phi|^2\,ds+2\int_\tau^t e^{ms}\left(|v|^2+|\psi|^2\right)ds\right)+\frac{e^{mt}}{m}K_3^2\\
&\qquad-\left(\nu-\frac{6C_1^4}{\nu\lambda_1}K_1^2-\frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}\right.\\
&\qquad\left.-\frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}-\frac{2C_2C_g}{3\nu\lambda_1}-\frac{m}{\lambda_1}-m\alpha^2\right)\int_\tau^t e^{ms}\left|A^{3/4}v\right|^2ds\\
&\quad\le\frac{e^{mt}}{m}K_3^2+\left(\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}+C_g\right)e^{m\tau}\|\phi\|^2_{L^2_H}+\frac{48ae^{mh}e^{m\tau}}{\nu\lambda_1^{1/2}(1-M)}|\psi|^2h+\frac{2C_g}{m}e^{mt}|\psi|^2\\
&\qquad-\left(\nu-\frac{6C_1^4}{\nu\lambda_1}K_1^2-\frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}-\frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}-\frac{2C_2C_g}{3\nu\lambda_1}-\frac{m}{\lambda_1}\right.\\
&\qquad\left.-m\alpha^2-\frac{24ae^{mh}}{\nu\lambda_1^2(1-M)}-\frac{2C_g}{\lambda_1^{\frac{3}{2}}}\right)\int_\tau^t e^{ms}\left|A^{3/4}v\right|^2ds\\
&\quad\le\frac{e^{mt}}{m}K_3^2+\left(\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}+C_g\right)e^{m\tau}\|\phi\|^2_{L^2_H}\\
&\qquad+\frac{48ae^{mh}e^{m\tau}}{\nu\lambda_1^{1/2}(1-M)}|\psi|^2h+\frac{2C_g}{m}e^{mt}|\psi|^2.
\end{aligned}
\tag{4.16}
$$

It follows that

$$\begin{aligned}
&\left|A^{1/4}v\right|^2+\alpha^2\left|A^{3/4}v\right|^2\\
&\quad\le \frac{K_3^2}{m}+\frac{2C_g}{m}|\psi|^2+\left|A^{1/4}v(\tau)\right|^2+\alpha^2\left|A^{3/4}v(\tau)\right|^2\\
&\qquad+\left(\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}+C_g\right)\|\phi\|^2_{L^2_H}+\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}|\psi|^2h\\
&\quad\le \frac{K_3^2}{m}+\frac{2C_gC_4^2\|\varphi\|^2_{L^\infty(\partial\Omega)}}{m}+d^2+\left(\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}+C_g\right)d^2_{L^2_H}\\
&\qquad+\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}C_4^2\|\varphi\|^2_{L^\infty(\partial\Omega)}h\equiv K_4^2.
\end{aligned}\tag{4.17}$$

Integrating (4.14) over $[t,t+1]$, we obtain

$$\begin{aligned}
&\left(\left|A^{1/4}v(t+1)\right|^2+\alpha^2\left|A^{3/4}v(t+1)\right|^2\right)-\left(\left|A^{1/4}v(t)\right|^2+\alpha^2\left|A^{3/4}v(t)\right|^2\right)\\
&\qquad+\left(\nu-\frac{6C_1^4}{\nu\lambda_1}K_1^2-\frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}-\frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|^2_{L^\infty(\partial\Omega)}-\frac{2C_2C_g}{3\nu\lambda_1}\right)\\
&\qquad\times\int_t^{t+1}\left|A^{3/4}v\right|^2ds\\
&\quad\le K_3^2+\frac{24a}{\nu\lambda_1^{1/2}}\int_t^{t+1}\left|v\big(s-\rho(s)\big)\right|^2ds+\frac{1}{C_g}\int_t^{t+1}|g|^2ds\\
&\quad\le K_3^2+\frac{24a}{\nu\lambda_1^{1/2}(1-M)}\int_{t-h}^{t+1}\left|v(s)\right|^2ds+C_g\int_{t-h}^{t+1}\left|v(s)+\psi\right|^2ds\\
&\quad\le K_3^2+\frac{24a}{\nu\lambda_1^{1/2}(1-M)}\int_{t-h}^{t+1}\left|v(s)\right|^2ds+2C_g\int_{t-h}^{t+1}\left|v(s)\right|^2ds+2C_g\int_{t-h}^{t+1}|\psi|^2ds\\
&\quad\le K_3^2+2(h+1)|\psi|^2C_g+\left(\frac{24a}{\nu\lambda_1^{1/2}(1-M)}+2C_g\right)\int_{t-h}^{t+1}\left|v(s)\right|^2ds\\
&\quad\le\left(\frac{24a}{\nu\lambda_1^{1/2}(1-M)}+2C_g\right)\left(\int_{t-h}^{t}\left|v(s)\right|^2ds+\int_t^{t+1}\left|v(s)\right|^2ds\right)\\
&\qquad+K_3^2+2(h+1)|\psi|^2C_g\\
&\quad\le\left(\frac{24a}{\nu\lambda_1^{1/2}(1-M)}+2C_g\right)\left(K_1^2h+\int_{\tau-h}^{\tau}|\eta|^2ds+\int_t^{t+1}\left|v(s)\right|^2ds\right)\\
&\qquad+K_3^2+2(h+1)|\psi|^2C_g\\
&\quad\le\left(\frac{24a}{\nu\lambda_1^{1/2}(1-M)}+2C_g\right)\left(K_1^2h+2\|\phi\|^2_{L^2_H}+2h|\psi|^2+\frac{1}{\lambda_1^{\frac{3}{2}}}\int_t^{t+1}\left|A^{\frac{3}{4}}v\right|^2ds\right)\\
&\qquad+K_3^2+2(h+1)|\psi|^2C_g\\
&\quad\le K_3^2+2(h+1)|\psi|^2C_g+\left(\frac{24a}{\nu\lambda_1^{1/2}(1-M)}+2C_g\right)\left(K_1^2h+2\|\phi\|^2_{L^2_H}+2h|\psi|^2\right)\\
&\qquad+\left(\frac{24a}{\nu\lambda_1^{2}(1-M)}+\frac{2C_g}{\lambda_1^{\frac{3}{2}}}\right)\int_t^{t+1}\left|A^{\frac{3}{4}}v\right|^2ds
\end{aligned}$$

$$\begin{aligned}
&\le \left(\frac{24a}{\nu\lambda_1^{1/2}(1-M)} + 2C_g\right)\left(K_1^2 h + 2\|\phi\|_{L_H^2}^2 + 2hC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2\right) \\
&\quad + \left(\frac{24a}{\nu\lambda_1^2(1-M)} + \frac{2C_g}{\lambda_1^{\frac{3}{2}}}\right)\int_t^{t+1}\left|A^{\frac{3}{4}}v\right|^2 ds + K_3^2 + 2(h+1)C_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2 C_g \qquad (4.18)
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^{t+1}\left|A^{3/4}v\right|^2 ds \\
&\quad \le \frac{N^2}{\nu - \frac{6C_1^4}{\nu\lambda_1}K_1^2 - \frac{6C_2^2C_4^2}{\nu\lambda_1}\|\varphi\|_{L^\infty(\partial\Omega)}^2 - \frac{6C_2^2C_3C_4^2}{\nu\lambda_1}\|\varphi\|_{L^\infty(\partial\Omega)}^2 - \frac{2C_2C_g}{3\nu\lambda_1} - \frac{24a}{\nu\lambda_1^2(1-M)} - \frac{2C_g}{\lambda_1^{\frac{3}{2}}}} \\
&\quad \equiv I_{3/4}^2, \qquad (4.19)
\end{aligned}$$

where

$$\begin{aligned}
N^2 &= \left(\frac{24a}{\nu\lambda_1^{1/2}(1-M)} + 2C_g\right)\left(K_1^2 h + 2\|\phi\|_{L_H^2}^2 + 2hC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2\right) + K_2^2 \\
&\quad + K_3^2 + 2(h+1)C_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2 C_g,
\end{aligned}$$

which implies $v \in L^\infty(\tau, T; D(A^{3/4})) \cap L^2(\tau, T; D(A^{3/4}))$.

From (4.16), we also have

$$\begin{aligned}
&e^{mt}\left(\left|A^{1/4}v\right|^2 + \alpha^2\left|A^{3/4}v\right|^2\right) - e^{m\tau}\left(\left|A^{1/4}v(0)\right|^2 + \alpha^2\left|A^{3/4}v(0)\right|^2\right) \\
&\quad \le \frac{K_3^2 + 2C_gC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{m}e^{mt} + \left(\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)} + C_g\right)e^{m\tau}\|\phi\|_{L_H^2}^2 \\
&\quad\quad + \frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}e^{m\tau}|\psi|^2 h \qquad (4.20)
\end{aligned}$$

and

$$\begin{aligned}
&\left|A^{1/4}v\right|^2 + \alpha^2\left|A^{3/4}v\right|^2 \\
&\quad \le e^{m(\tau-t)}\left(\left(1 + \frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)} + C_g\right)d^2 + \frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}C_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2 h\right) \\
&\quad\quad + \frac{K_3^2 + 2C_gC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{m}. \qquad (4.21)
\end{aligned}$$

Choosing $\sigma \in [-h, 0]$ and substituting t with $t+\sigma$, we have

$$\begin{aligned}
&\left|A^{1/4}v(t+\sigma)\right|^2 + \alpha^2\left|A^{3/4}v(t+\sigma)\right|^2 \\
&\quad \le e^{mh}e^{m(\tau-t)}\left(\left(1 + \frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)} + C_g\right)d^2 + \frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}C_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2 h\right) \\
&\quad\quad + \frac{K_3^2 + 2C_gC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{m}. \qquad (4.22)
\end{aligned}$$

Denoting $v_t(\cdot;(v_\tau,\phi))$ as $\tilde{U}(\cdot;\tau,(v_\tau,\phi))$, we get

$$\begin{aligned}\|v_t\|_{3/4}^2 &= \big\|\tilde{U}(\cdot;\tau,(v_\tau,\phi))\big\|_{3/4}^2\\ &\le e^{mh}e^{m(\tau-t)}\left(\left(1+\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}+C_g\right)d^2+\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}C_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2h\right)\\ &\quad+\frac{K_3^2+2C_gC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{m}. \end{aligned}\tag{4.23}$$

Let $\rho_{3/4}^2=\frac{2K_3^2+4C_gC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{m}$ for any $(v_\tau,\eta)\in D(A^{3/4})\times L_H^2$, when

$$\tau\le T_{3/4}=\frac{1}{m}\ln\frac{K_3^2+2C_gC_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2}{m((1+\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}+C_g)d^2+\frac{48ae^{mh}}{\nu\lambda_1^{1/2}(1-M)}C_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2h)}-h+t,$$

$\tilde{U}(\cdot,\cdot)D\subset B_{3/4}(0,\rho_{3/4})$, where $B_{3/4}(0,\rho_{3/4})$ is a pullback absorbing ball centered at 0 with radius $\rho_{3/4}$ in $C_{3/4}$, which completes the proof. □

5 Existence of pullback attractor

The main result in the paper can be stated as follows.

Theorem 5.1 *Assume that the assumptions* (a)-(h) *hold*, $(v_\tau,\eta)\in D(A^{\frac{3}{4}})\times L_H^2$, *then the system* (1.1) *possesses a pullback attractor* $\mathcal{A}$.

Proof Theorem 3.1 guarantees that the process $\{\tilde{U}(\cdot,\cdot)\}$ of the system (3.15) is continuous. By Theorem 4.1 and Theorem 4.2 we show that the system (3.15) possesses two pullback absorbing balls $B_V(0,\rho_V)$ and $B_{3/4}(0,\rho_{3/4})$ in C_V and $C_{D(A^{3/4})}$, respectively. Since $V\hookrightarrow\hookrightarrow D(A^{3/4})$ and $\{\tilde{U}(\cdot,\cdot)\}$ is equicontinuous, by the generalized Arzelà-Ascoli theorem we can show that the process $\{\tilde{U}(\cdot,\cdot)\}$ is asymptotically compact in C_V. From the fundamental existence theory of pullback attractors (see, *e.g.*, [21, 22]), the process $\{\tilde{U}(\cdot,\cdot)\}$ generated by the system (3.15), which is equivalent to (1.1), possesses a pullback attractor in C_V. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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